Mixed analytical/numerical method applied to the low Reynolds number k-epsilon turbulence model

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Abstract

A mixed analytical/numerical method is developed here to solve the low Reynolds number k-epsilon turbulence model. In this method the advection-diffusion part is solved numerically, while the source terms are split into two parts: one part is solved analytically and the next is solved numerically.

Keyword: *Mixed analytical/numerical method, k-epsilon turbulence model,low Reynolds number,, source term*

1. Introduction

Because of the recent enormously progress in the capability of computers, low Reynolds two-equation turbulent models become more and more welcome in engineering fluid computation. In the past, various forms of low-Reynolds-number k - ε turbulent models have been proposed. The detail of two-equation models and low Reynolds corrections are presented in Chapter 14 (Turbulence Modeling and Simulation) of handbook [1].

Though mathematically the $k-\varepsilon$ model is well-posed[2], the strong nonlinearities may interact with numerical errors in such a way that computation may break down easily.. A typical behavior of unstable computations involves the loss of positivity of k or ε, though the original differential equations have positive solution [3]. The appearance of negative values changes the sign of several terms in the models, so that turbulent quantities may increase unboundedly [4]. Even though the two turbulence equations can be solved exactly in the same manner as the mean flow equations, it has been found that such a method often leads to an unstable solution, or even incorrect solutions [5]. The damping functions in the low Reynolds turbulent models improve the model prediction capability for near wall flow, but also introduce more severe numerical stiffness for the source terms.

There are a huge amount of numerical methods for compressible Navier-Stokes equations coupled with two equation models [6-9]. The two-equation turbulent model is a typical example of partial differential equations with source terms. Great progress has been made in efficient treatment of the source terms [10-12]. Helzel, LeVeque, and Warnecke [12] treated chemical reacting flow with an Arrhenius law for the source term by mixed method in detonation waves computation. In [13], a mixed analytical/numerical method for oscillating source terms has studied. In this method the advection-diffusion part and the source terms are treated separately through operator splitting. The advection-diffusion part (PDE) is integrated numerically while the source term part (ODE) is integrated analytically. Hence this method is called mixed analytical/numerical method. The mixed method performs well for partial differential equations with source terms, in which the time scale of source term (S-scale, denoted T_S) is much smaller than the mean flow scale (M-Scale, denoted T_M) inherent to the advection-diffusion part. Furthermore, the mixed method has been extended to the implicit solution of high Reynolds number and compressible turbulent flows[14]. Numerical results show the mixed method can give robust, steady and fast convergent solution.

In this paper, we extend the mixed method to low Reynold number turbulent models. The essential new feature of the mixed method for low Reynolds number turbulent models lies in the treatment of the source terms which contain new damping functions. With respect to the high Reynolds number counterpart, the low Reynolds number turbulent models retain the

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE **15 APR 2005**	2. REPORT TYPE **N/A**	3. DATES COVERED **-**

4. TITLE AND SUBTITLE **Mixed analytical/numerical method applied to the low Reynolds number k-epsilon turbulence model**	5a. CONTRACT NUMBER 5b. GRANT NUMBER 5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER 5e. TASK NUMBER 5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Department of Engineering Mechanics, Tsinghua University, Beijing 100084, P.R. China**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S) 11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release, distribution unlimited

13. SUPPLEMENTARY NOTES
See also ADM001800, Asian Computational Fluid Dynamics Conference (5th) Held in Busan, Korea on October 27-30, 2003., The original document contains color images.

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**UU**	**7**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

orginal source terms modified with damping factors and sometimes contain additional damping terms. In the mixed method proposed in this paper, the damping factors are treated as constant at each time step, so that the main part of the source terms are still analytically integrable at each time step. The additional damping terms are treated numerically.

2. Governing Equations

The governing equations are obtained by Favre Averaging the Navier-Stokes equations and modeling the Reynolds stress. In conservative form these equations are written as

$$\frac{\partial U}{\partial t}+\frac{\partial F_c}{\partial x}+\frac{\partial G_c}{\partial y}=\frac{\partial F_v}{\partial x}+\frac{\partial G_v}{\partial y} \tag{1}$$

In this paper, we use the Hwang-Lin low-Reynolds number k-$\tilde{\varepsilon}$ turbulence model[14] as example.

$$\frac{\partial U^T}{\partial t}+\frac{\partial F_c^T}{\partial x}+\frac{\partial G_c^T}{\partial y}=\frac{\partial F_v^T}{\partial x}+\frac{\partial G_v^T}{\partial y}+S \tag{2}$$

where

$$U^T=\begin{bmatrix}\rho k\\ \rho\tilde{\varepsilon}\end{bmatrix}\quad F_c^T=\begin{bmatrix}\rho uk\\ \rho u\tilde{\varepsilon}\end{bmatrix}\quad G_c^T=\begin{bmatrix}\rho vk\\ \rho v\tilde{\varepsilon}\end{bmatrix}, \tag{3}$$

$$F_v^T=\begin{bmatrix}\left(\mu+\frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x}\\ \left(\mu+\frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial\tilde{\varepsilon}}{\partial x}\end{bmatrix}\qquad G_v^T=\begin{bmatrix}\left(\mu+\frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\\ \left(\mu+\frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial\tilde{\varepsilon}}{\partial y}\end{bmatrix} \tag{4}$$

and

$$S=\begin{bmatrix}P_k-\rho\varepsilon+\Pi\\ f_1C_{\varepsilon1}\frac{\tilde{\varepsilon}}{k}P_k-f_2C_{\varepsilon2}\frac{\rho\tilde{\varepsilon}^2}{k}+\xi\end{bmatrix} \tag{5}$$

The production term P_k is given by

$$P_k=\mu_t\left(\frac{\partial U_i}{\partial x_k}+\frac{\partial U_k}{\partial x_i}\right)\frac{\partial U_i}{\partial x_k} \tag{6}$$

The eddy viscosity is calculated as

$$\mu_t=f_\mu C_\mu\rho k^2/\tilde{\varepsilon} \tag{12}$$

The functions Π and ξ are defined as

$$\Pi=-\frac{1}{2}\frac{\partial}{\partial x_j}\left(\nu\frac{k}{\varepsilon}\frac{\partial D}{\partial x_j}\right),\quad \xi=-\frac{1}{2}\frac{\partial}{\partial x_j}\left(\nu\frac{\tilde{\varepsilon}}{k}\frac{\partial k}{\partial x_j}\right) \tag{13}$$

The coefficients and damping functions are

$$C_\mu=0.09,\ \ C_{\varepsilon1}=1.44,\ \ C_{\varepsilon2}=1.92$$

$$f_\mu=1-\exp[-0.01y_\lambda-0.008y_\lambda^3]$$

$$\sigma_k=1-\exp[-0.01y_\lambda-0.008y_\lambda^3],\ \sigma_\varepsilon=1.3-1.0\exp[-y_\lambda/10]$$

$$f_1=1.0,\ \ f_2=1.0$$

$$\varepsilon=\tilde{\varepsilon}+2\nu\left(\frac{\partial\sqrt{k}}{\partial y}\right)^2$$

where $y_\lambda={}^{y}\!\big/\!_{\sqrt{\nu k/\tilde{\varepsilon}}}$.

3. Numerical Method

The two set equations (1) and (2) are solved separately. The convective numerical flux at the cell interface is evaluated using Roe's approximated Riemann solver with MUSCL treatment to achieve second order accuracy. The simplified multistage scheme with four stages is applied to advance in time.

The advection-diffusion part of the two-equation model is discretized in the same way as for the Navier-Stokes equations. Only the source term needs special treatment. The numerical treatment of source terms of the turbulence model equations is of mayor importance for the stability of this scheme[16]. The fundamental means is to treat the negative source terms implicitly and the positive terms explicitly[7]. However, the time step still has to be small enough in order to obtain realistic values of k, ε and μ_t. According to the different treatment for Jacobian matrix for implicit part, there are three method: point implicit method[6], approximate Jacobian method[5,17] and exact Jacobian method[18]. In this paper, we adopt the second method.

4. Mixed Analytical/numerical Method for low Reynolds number turbulent models

The turbulence model equations contain advection-diffusion operators and source terms. In the mixed method the advection diffusion part (PDE) is integrated numerically, while the source term (ODE) is integrated analytically.

4.1 Treatment of the source terms

For the case of low Reynolds number models it is not always possible to integrate the source terms analytically. With respect to the high Reynolds number counterpart, the low Reynolds number turbulent models retain the original source terms modified with damping factors and sometimes contain additional damping terms. In the mixed method proposed in this paper, the damping factors are treated as constant at each time step, so that the main part of the source terms are still analytically integrable at each time step. The additional damping terms are treated numerically.

Following the original construction of the mixed method, we first consider the ODE due to the source terms

$$\frac{dU^T}{dt} = S_s(U) + S_D(U) \tag{7}$$

Here $S_S(U)$ is the standard part which is similar to the source term of the high Reynolds number model, modified only by adding damping factors, and $S_D(U)$ stands for the additional damping terms. Precisely, $S_S(U)$ is defined by

$$S_s(U) = \begin{pmatrix} P_k - \rho\tilde{\varepsilon} \\ f_1 C_{\varepsilon 1} \frac{\tilde{\varepsilon}}{k} P_k - f_2 C_{\varepsilon 2} \frac{\rho\tilde{\varepsilon}^2}{k} \end{pmatrix} \tag{8}$$

The additional damping terms $S_D(U)$ are treated numerically, while the standard part $S_S(U)$ is solved analytically by considering the following ODE

$$\begin{cases} \dfrac{d\rho k}{dt} = P_k - \rho\tilde{\varepsilon} \\ \dfrac{d\rho\tilde{\varepsilon}}{dt} = C_{\varepsilon 1}{}' P_k \frac{\tilde{\varepsilon}}{k} - C_{\varepsilon 2}{}' \frac{\rho\tilde{\varepsilon}^2}{k} \end{cases} \tag{9}$$

with the parameters

$$C_{\varepsilon 1}{}' = f_1 C_{\varepsilon 1} \qquad C_{\varepsilon 2}{}' = f_2 C_{\varepsilon 2} \quad C_\mu{}' = f_\mu C_\mu$$

treated as constant at each time step. Hence the standard part should have the same form of analytical solution as the high Reynolds number model[13], and this analytical solution for (16) can be written as

$$\rho k = K\left((\rho k)_0, (\rho\tilde{\varepsilon})_0, t\right) = (\rho k)_0 \, 2^\alpha F^\gamma \left(\frac{G}{\frac{1}{\zeta}\Psi_0} \right)^{-\beta} \exp(-\delta\sqrt{\Phi} t) \tag{10}$$

$$\rho\tilde{\varepsilon} = \mathrm{E}\left((\rho k)_0, (\rho\tilde{\varepsilon})_0, t\right) = (\rho\tilde{\varepsilon})_0 \, 2^\alpha F^\upsilon \left(\frac{G}{\frac{1}{\zeta}\Psi_0} \right)^{-\tau} \exp(-\delta\sqrt{\Phi} t) \tag{11}$$

where

$$F = 1 - \frac{1}{\zeta}\Psi_0 + (1 + \frac{1}{\zeta}\Psi_0)\exp\left(2\mu\sqrt{\Phi}t\right)$$

$$G = \frac{1}{\zeta}\Psi_0 - 1 + (1 + \frac{1}{\zeta}\Psi_0)\exp\left(2\mu\sqrt{\Phi}t\right)$$

$$\Psi_0 = \frac{k_0}{\varepsilon_0}\Phi^{\frac{1}{2}}$$

$$\aleph = \sqrt{\left(C'_{\varepsilon 2}-1\right)C'_\mu\left(C'_{\varepsilon 1}-1\right)},\ \zeta = \sqrt{\frac{C'_{\varepsilon 2}-1}{C'_\mu\left(C'_{\varepsilon 2}-1\right)}},\ \beta = \frac{1}{C'_{\varepsilon 2}-1}$$

$$\alpha = \frac{C'_{\varepsilon 1}-C'_{\varepsilon 2}}{\left(C'_{\varepsilon 2}-1\right)\left(C'_{\varepsilon 1}-1\right)},\quad \zeta = \frac{1}{C'_{\varepsilon 1}-1},\ \sigma = \frac{C'_\mu\left(C'_{\varepsilon 2}-C'_{\varepsilon 1}\right)}{\sqrt{\left(C'_{\varepsilon 2}-1\right)C'_\mu\left(C'_{\varepsilon 1}-1\right)}}$$

$$\delta = \frac{C'_\mu\left(C'_{\varepsilon 2}-C'_{\varepsilon 1}\right)}{\sqrt{\left(C'_{\varepsilon 2}-1\right)C'_\mu\left(C'_{\varepsilon 1}-1\right)}},\qquad \varpi = \frac{C'_{\varepsilon 2}}{C'_{\varepsilon 2}-1}$$

$$\varsigma = \frac{C'_{\varepsilon 1}-C'_{\varepsilon 2}}{\left(C'_{\varepsilon 2}-1\right)\left(C'_{\varepsilon 1}-1\right)},\qquad \upsilon = \frac{C_{\varepsilon 1}'}{C_{\varepsilon 1}'-1}$$

Here k_0 and ε_0 stand for initial value of turbulent kinetic energy k and dissipation rate ε.

4.2 Mixed method

Now the mixed method for low Reynolds number models can be written as

$$\left(I - \Delta t \cdot [A]^n\right)\Delta\left(U^T\right)^n = \Delta t \cdot R_j^n + \bar{S}_S\left(\left(U^T\right)^n\right) + S_D\left(\left(U^T\right)^n\right) \tag{12}$$

where $S_D\left(\left(U^T\right)^n\right)$ is due to the additional damping terms, and $\bar{S}_S\left(\left(U^T\right)^n\right)$ is the analytical solution of the ODE due to the standard source terms. This analytical solution, when limited, can be written as

$$\bar{S}_S\left(\left(U^T\right)^n\right) = \begin{bmatrix} \min\left(K\left((\rho k)_j^n,(\rho\varepsilon)_j^n,\Delta t\right)-(\rho k)_j^n,\ \ \delta\right) \\ \min\left(\mathrm{E}\left((\rho k)_j^n,(\rho\varepsilon)_j^n,\Delta t\right)-(\rho\varepsilon)_j^n,\ \ \delta\right) \end{bmatrix} \tag{13}$$

The parameter δ limits the maximum of the source term integral. The choice of δ=0.1 works well in our numerical experiments.

5. Numerical Results

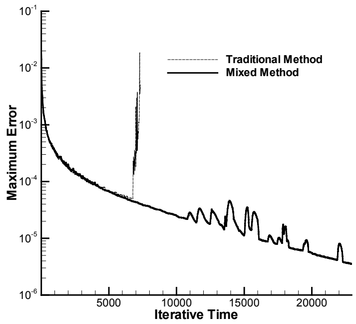
5.1 Flat plate

We first calculate a turbulent flow past a flat plane. The grid is 80×41. The distance of the first grid next to the wall is no more than $y^+=0.45$. The Reynolds number is $Re=1.0\times10^7$, the inflow Mach number is Ma=0.2. The convergence history of mixed method is compared with that of traditional method in Fig.1. The latter can't converge with the same condition. The inner layer velocity profile of flat plane flow is shown in Fig.2, comparing the van Driest's and Spalding's theory. The figure gives the velocity profile of 7 sites in x direction. The agreement is also good.

5.2 Transonic Diffuser

Consider transonic flow with a weak shock through a converging diverging diffuser[19]. This configuration has an entrance to throat area ratio of 1.4, an exit to throat area ratio of 1.5. The corresponding Reynolds number is 9.370×10^5 and the Mach number is $M_{in}=0.9$. The total pressure at inflow is 1.349×10^5 Pa, the static pressure at the outflow is 1.11×10^5Pa. These flows were characterized by the ratio, R, of exit static to inflow total pressure. For the weak-shock case the value of R was 0.82.

The convergence history is shown in Fig.3, comparing that with traditional method. We see the computation with traditional method fails to converge, while the mixed method not only converge steadily, but also converge faster than the traditional method.



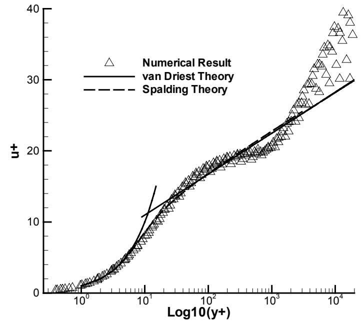
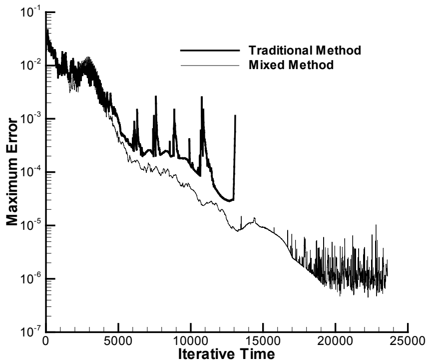


Fig.1 Fig.2

Fig.1. Comparative convergence history for flat plane

Fig.2. Velocity profile of flat plate flow



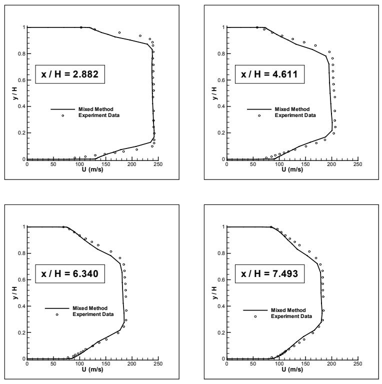


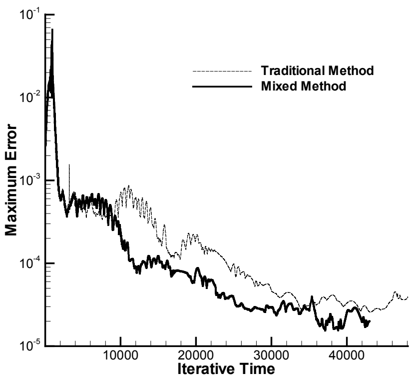
Fig.3 Fig.4

Fig.3. Comparative convergence history for transonic diffuser

Fig.4 Velocity profiles at four axial locations for transonic diffuser

5.3 RAE2822 Airfoil

The RAE2822 airfoil has been extensively used for the validation of Navier-Stokes codes applied to transonic flow. The key flow condition for this test case is: Free stream Mach Number Ma∞=0.73, Reynolds number $Re=6.5\times10^{6}$, and angle of attack in degree α=3.19. The convergence histories with mixed method and with conventional numerical method are compared in Fig.5. The first one converges faster. The pressure coefficient and skin friction coefficient with experimental results are shown in Fig.6 and Fig.7 respectively, comparing the conventional method results. The numerical results with mixed method agree with experiment data.



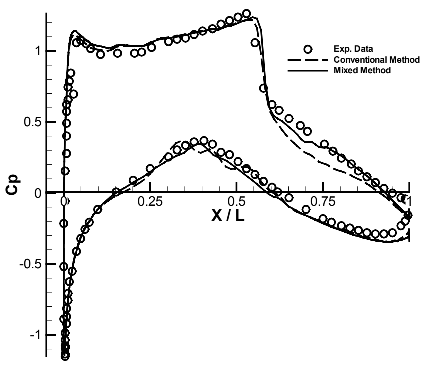


Fig. 5 Fig.6

Fig. 5. Comparative convergence history for RAE2822 airfoil

Fig. 6. The pressure coefficient profiles of RAE2822 airfoil

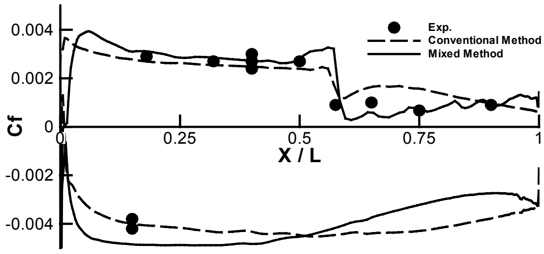


Fig. 7. The skin friction coefficient profiles of RAE2822 airfoil

5.4 NLR7301 Wing-flap Airfoil

The flow around the two-element NLR7301 airflow with 2.6% gap has been computed for the flow condition: Re=2.51×106 and angle of attack α=13.1. Fig.8 gives the pressure distribution on wing and flap.

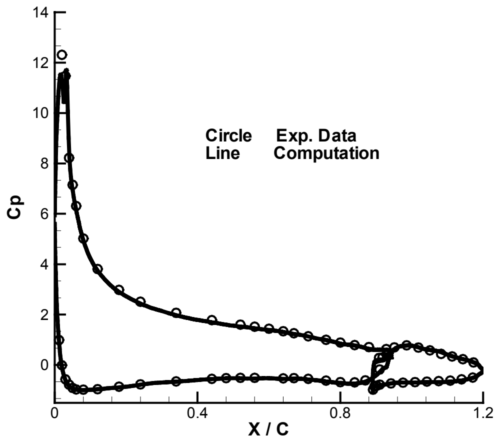


Fig. 8. The pressure coefficient profiles of NLR7301 airfoil

6. Concluding Remarks

The mixed analytical/numerical method has been extended to the numerical solutions of low Reynolds number k-ε turbulence models. The mixed method is applied to Hwang-Lin low Reynolds turbulent model and several test problems. The numerical results show that the mixed method is numerically more robust than the traditional pure numerical method.

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